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New Approach to the Maneuvering and Control of Flexible Multibody Systems

Moon K. Kwak* and Leonard Meirovitch†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

A mathematical formulation capable of treating the problem of maneuvering and control of flexible multibody systems is developed. The formulation is based on equations of motion in terms of quasicoordinates derived independently for the individual substructures and on a consistent kinematical synthesis causing the substructures to act as a single structure. A perturbation approach permits the separation of the nonlinear high-dimensional system of equations into a zero-order, low-dimensional problem for the rigid-body maneuvering and a first-order, high-dimensional, time-varying problem for the elastic motions and the perturbations from the rigid-body motions. The formulation lends itself to ready computer implementation. A numerical example involving a three-beam system demonstrates the effectiveness of the new algorithm.

I. Introduction

FOR a number of years there has been a persistent interest in the dynamics of flexible multibody systems. This type of system is encountered in flexible robots, rotorcraft, and spacecraft. Equations of motion for flexible multibody systems have been derived by a variety of approaches.[1-9]

Recently, the interest has broadened so as to include maneuvering and control. This narrows the choice of formulations significantly, as the formulation must be consistent with the control task. A set of equations of motion suited for the control task can be formulated by means of Lagrangian equations for flexible bodies in terms of quasicoordinates.[10] The advantage of this approach is that it yields equations in terms of body axes, which are the same axes as those used to express control forces and torques. In using the approach of Ref. 10 to derive equations of motion for a chain of flexible multibody systems, it is convenient to adopt a kinematic procedure permitting the expression of the velocity vector of a nominal point in a typical body in terms of the velocity vector of the preceding body in the chain. The resulting differential equations are nonlinear and hybrid,[11] where the term "hybrid" implies that the equations for the rigid-body translations and rotations are ordinary differential equations and those for the elastic motions are partial differential equations. Because maneuvering and control design in terms of hybrid equations is not feasible, the partial differential equations must be transformed into sets of ordinary differential equations by means of a discretization-in-space procedure, such as the finite element method[12] or a Rayleigh-Ritz-based substructure synthesis.[13] The resulting formulation consists of a high-order set of nonlinear ordinary differential equations. A common approach to control design requires the solution of a two-point boundary-value problem. However, this is not feasible for high-order systems; hence, a different approach is advisable.

The nonlinearity enters into the differential equations through the rigid-body motions. Indeed, the elastic motions tend to be small. In view of this, it appears natural to conceive of a perturbation approach whereby the rigid-body motions can be regarded as being of zero order in magnitude and the elastic motions as being of first order in magnitude. This approach permits dividing the problem into a low-dimensional set of nonlinear zero-order equations for the rigid-body motions and a high-dimensional set of linear first-order equations for the elastic motions and the perturbations in the rigid-body motion, where the order is to be taken in a perturbation sense. Note that, because the zero-order solution enters into the first-order equations as a known function of time, the first-order equations represent a time-varying system. Moreover, the system is subjected to persistent disturbances. The perturbation approach just described, first proposed in Ref. 14, was used in Refs. 15-19 to maneuver and control flexible spacecraft.

The kinematical synthesis of Refs. 14-19 works quite well in the case in which the number of bodies in the chain is relatively small. When the number of bodies is larger than three, difficulties can be expected, so that a different approach is advisable. In Refs. 14-19, the kinematical synthesis was implemented before the derivation of the equations of motion was carried out. In this paper, we consider a procedure whereby the equations of motion are derived first for each individual flexible body. Then the sets of equations for the individual bodies are assembled into a global set by invoking the kinematical relations described earlier. In the process, the redundant coordinates and velocities resulting from considering the individual bodies separately are eliminated. It is convenient to carry out the kinematical synthesis on the zero-order problem and first-order problem separately. Implementation of the kinematical synthesis is based on recursive relations that lend themselves to ready computer coding. The resulting zero- and first-order global sets of equations are particularly suited for maneuvering and control design, respectively.

The zero-order nonlinear equations govern the maneuver as if the system consisted of articulated rigid bodies where the maneuver amounts to driving the system from an initial state to a final state. The equations can be solved open loop or closed loop. For minimum-time maneuvering, the control law is bang-bang. The simplest approach is to carry out the maneuver by means of actuators that impart predetermined motions to the substructures relative to one another. The first-order equations govern the elastic vibrations and the perturbations in the rigid-body motions. They contain the zero-order solution as a known function of time. As a result, the system is time varying. Moreover, it is subjected to persistent disturbances caused by the maneuver. The process can be likened to that in which the system must follow a reference state. In this case the reference state is defined by the rigid-body maneuvering, which is characterized by zero elastic states. Then the first-order equations are simply the equations in terms of the

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*Assistant Professor, Department of Engineering Science and Mechanics. Member AIAA.

†University Distinguished Professor, Department of Engineering Science and Mechanics. Fellow AIAA.

difference between the actual states and the reference states, where this difference can be identified as perturbations in the state variables. Control of the first-order system is carried out closed loop and includes a disturbance accommodation procedure. The latter depends on the type of disturbance, which in turn depends on the type of maneuver performed.

As a numerical example, a three-beam system is considered, in which a bang-bang control torque is applied to the center beam. Numerical results show the effectiveness of the new algorithm developed in this paper.

II. Hybrid Equations for the Substructures

Our interest lies in deriving equations of motion capable of describing the problem of maneuvering and control of structures in the form of an articulated chain of N flexible substructures. Such equations were derived in Refs. 11 and 16–19 by means of Lagrangian equations for flexible bodies in terms of quasicoordinates.[10] To this end, extensive use was made of a kinematical procedure whereby the motion was referred to sets of local body axes embedded in the undeformed substructures and the motion of one substructure was described in terms of the motion of the preceding substructure in the chain. This kinematical procedure obviates the need for constraint equations. The approach of Refs. 11 and 16–19 was made easier by the assumption that, for maneuvering substructures, the relative motion between any two substructures was prescribed a priori and could be regarded as known.

In this paper we consider the case in which the relative motion between any two substructures is not prescribed a priori. The kinematical procedure described previously can be used here as well. However, when the number N of substructures in the chain becomes very large, the equations of motion become very complicated, so that the question arises as to alternative approaches. One such approach is to derive equations of motion for the individual substructures separately and then impose kinematical relations of the type described earlier to obtain the system's equations of motion. This is the alternative proposed in this paper.

Let us consider a typical substructure s and introduce the inertial axes $X_I Y_I Z_I$ with the origin at I and a set of body axes $x_s y_s z_s$ with the origin at S and embedded in the undeformed substructure (Fig. 1). Then we can write the position vector of a typical point in s with the spatial coordinates given symbolically by P_s as follows:

$$\boldsymbol{W}_s(t) = \boldsymbol{R}_s(t) + \boldsymbol{r}_s(P_s) + \boldsymbol{u}_s(P_s,t), \quad s = 1,2,\ldots,N \tag{1}$$

where $\boldsymbol{R}_s$ is the radius vector from I to S, $\boldsymbol{r}_s$ is the radius vector from S to a typical point in s, and $\boldsymbol{u}_s$ is the elastic displacement vector of the same point relative to the body axes $x_s y_s z_s$ (Fig. 1). Note that $\boldsymbol{R}_s$ is in terms of inertial coordinates and $\boldsymbol{r}_s$ and $\boldsymbol{u}_s$ are in terms of components along the body axes. We propose to derive the equations of motion by the approach of Ref. 10. In fact, the equations for the individual bodies are identical to the hybrid equations derived there, so that here we merely present the pertinent material.

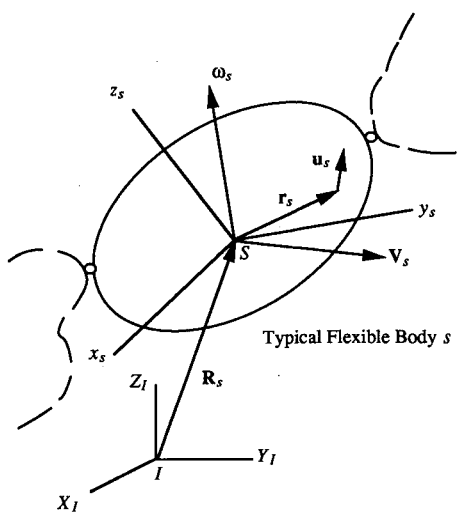

Fig. 1 Typical flexible body in the chain.

The Lagrangian formulation requires the kinetic energy, which in turn requires velocities. Assuming that axes $x_s y_s z_s$ rotate with the angular velocity $\boldsymbol{\omega}_s$ relative to the inertial space, the velocity vector of a typical point in s can be shown to be

$$\begin{aligned}\dot{\boldsymbol{W}}_s(t) &= \boldsymbol{V}_s(t) + \tilde{\boldsymbol{\omega}}_s(t)\left[\boldsymbol{r}_s(P_s) + \boldsymbol{u}_s(P_s,t)\right] + \boldsymbol{v}_s(P_s,t) \\ &= \boldsymbol{V}_s(t) + \left[\tilde{\boldsymbol{r}}_s(P_s) + \tilde{\boldsymbol{u}}_s(P_s,t)\right]^T \boldsymbol{\omega}_s(t) + \boldsymbol{v}_s(P_s,t) \\ & \qquad s = 1,2,\ldots,N \end{aligned} \tag{2}$$

where $\boldsymbol{V}_s$ is the velocity vector of S and $\boldsymbol{v}_s = \dot{\boldsymbol{u}}_s$ is the elastic velocity vector of the typical point. We note that a tilde over a symbol denotes a skew symmetric matrix formed from the corresponding vector.[10]

The vector $\boldsymbol{R}_s$ describes the position of the origin S of the body axes $x_s y_s z_s$ relative to the inertial axes $X_I Y_I Z_I$ in terms of inertial components. The orientation of the body axes $x_s y_s z_s$ relative to the inertial space is commonly described by a set of three angles, say, θ_{s1}, θ_{s2}, θ_{s3}. Euler's angles represent one of the possible sets. The time derivatives of these angles, $\dot{\theta}_{s1}$, $\dot{\theta}_{s2}$, $\dot{\theta}_{s3}$, can be interpreted as angular velocity components of the body about nonorthogonal axes, with one of the components having a fixed direction in the inertial space. On the other hand, the vectors $\boldsymbol{V}_s$ and $\boldsymbol{\omega}_s$ represent translational and rotational velocity vectors of the body axes, respectively, and are in terms of components about the orthogonal body axes. The relation between the velocity vectors in terms of components along body axes and inertial axes can be written as[10]

$$\boldsymbol{V}_s = \boldsymbol{C}_s \dot{\boldsymbol{R}}_s \tag{3a}$$

$$\boldsymbol{\omega}_s = \boldsymbol{D}_s \dot{\boldsymbol{\theta}}_s \tag{3b}$$

where $\boldsymbol{C}_s = \boldsymbol{C}_s(\theta_{s1},\theta_{s2},\theta_{s3})$ is the matrix of direction cosines between $x_s y_s z_s$ and $X_I Y_I Z_I$ and $\boldsymbol{D}_s = \boldsymbol{D}_s(\theta_{s1},\theta_{s2},\theta_{s3})$ is a transformation matrix. Note that the components of $\boldsymbol{V}_s$ and $\boldsymbol{\omega}_s$ can be regarded as time derivatives of quasicoordinates.[10] For various applications, including the one under consideration, it is advantageous to work with body axes components of motion, which makes the quasicoordinates a natural choice.

Next, we wish to present the equations of motion in terms of quasicoordinates for a single substructure. Using Eqs. (2), we can write the kinetic energy for substructure s in the following form:

$$\begin{aligned} T_s &= \frac{1}{2}\int_{\mathcal{D}_s} \rho_s \dot{\boldsymbol{W}}_s^T \dot{\boldsymbol{W}}_s \, d\mathcal{D}_s = \frac{1}{2} m_s \boldsymbol{V}_s^T \boldsymbol{V}_s + \boldsymbol{V}_s^T \tilde{\boldsymbol{S}}_s^T \boldsymbol{\omega}_s \\ &\quad + \boldsymbol{V}_s^T \int_{\mathcal{D}_s} \rho_s \boldsymbol{v}_s \, d\mathcal{D}_s + \frac{1}{2}\boldsymbol{\omega}_s^T \boldsymbol{J}_s \boldsymbol{\omega}_s \\ &\quad + \boldsymbol{\omega}_s^T \int_{\mathcal{D}_s} \rho_s (\tilde{\boldsymbol{r}}_s + \tilde{\boldsymbol{u}}_s)\boldsymbol{v}_s \, d\mathcal{D}_s + \frac{1}{2}\int_{\mathcal{D}_s} \rho_s \boldsymbol{v}_s^T \boldsymbol{v}_s \, d\mathcal{D}_s \end{aligned} \tag{4}$$

where ρ_s is the mass density, $\mathcal{D}_s$ is the domain of the substructure, and

$$m_s = \int_{\mathcal{D}_s} \rho_s \, d\mathcal{D}_s \tag{5a}$$

$$\tilde{\boldsymbol{S}}_s = \int_{\mathcal{D}_s} \rho_s (\tilde{\boldsymbol{r}}_s + \tilde{\boldsymbol{u}}_s) \, d\mathcal{D}_s \tag{5b}$$

$$\boldsymbol{J}_s = \int_{\mathcal{D}_s} \rho_s (\tilde{\boldsymbol{r}}_s + \tilde{\boldsymbol{u}}_s)(\tilde{\boldsymbol{r}}_s + \tilde{\boldsymbol{u}}_s)^T \, d\mathcal{D}_s \tag{5c}$$

Moreover, the potential energy can be expressed in the following form:

$$V_s = \tfrac{1}{2}[\boldsymbol{u}_s, \boldsymbol{u}_s] + V_{gs} \tag{6}$$

where [,] denotes symbolically an energy integral[12] and is a measure of the strain energy, and V_{gs} is a gravitational potential energy.

From Ref. 10 we obtain the general hybrid Lagrangian equations in terms of quasicoordinates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_s}{\partial \boldsymbol{V}_s}\right) + \tilde{\omega}_s\left(\frac{\partial L_s}{\partial \boldsymbol{V}_s}\right) - C_s\left(\frac{\partial L_s}{\partial \boldsymbol{R}_s}\right) = \boldsymbol{F}_s \tag{7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_s}{\partial \boldsymbol{\omega}_s}\right) + \tilde{V}_s\left(\frac{\partial L_s}{\partial \boldsymbol{V}_s}\right) + \tilde{\omega}_s\left(\frac{\partial L_s}{\partial \boldsymbol{\omega}_s}\right) - (D_s^T)^{-1}\left(\frac{\partial L_s}{\partial \boldsymbol{\theta}_s}\right) = \boldsymbol{M}_s \tag{7b}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_s}{\partial \boldsymbol{v}_s}\right) - \left(\frac{\partial \hat{T}_s}{\partial \boldsymbol{u}_s}\right) + \mathcal{L}_s \boldsymbol{u}_s = \hat{\boldsymbol{U}}_s \tag{7c}$$

where $L_s = T_s - V_s$ is the Lagrangian, $\hat{L}_s$ the Lagrangian density, $\hat{T}_s$ the kinetic energy density and $\mathcal{L}_s$ a differential operator matrix whose entries depend on the nature of the elastic members,[12] and we observe that, for large angles, $\boldsymbol{\theta}_s$ is a vector only in a symbolic way. Moreover, $\boldsymbol{F}_s$ is a resultant force vector acting on s, $\boldsymbol{M}_s$ a resultant moment vector, and $\hat{\boldsymbol{U}}_s$ a force density vector. They represent nonconservative forces and torques and include control forces and torques. We note that Eqs. (7) are hybrid, as Eqs. (7a) and (7b) are ordinary differential equations for the rigid-body translations and rotations of the body axes $x_s y_s z_s$, respectively, and Eq. (7c) is a partial differential equation for the elastic displacements of a typical point in the substructure relative to axes $x_s y_s z_s$. We note that the operator $\mathcal{L}_s$ is related to the energy integral by

$$[\boldsymbol{u}_s, \boldsymbol{u}_s] = \int_{\mathcal{D}_s} \boldsymbol{u}_s^T \mathcal{L}_s \boldsymbol{u}_s \, \mathrm{d}\mathcal{D}_s \tag{8}$$

Equation (8) can be derived through integrations by parts.[12] The same integration by parts also yields boundary conditions involving geometric compatibility and force and moment balance. The boundary conditions go with Eq. (7c). Inserting Eq. (4) into Eqs. (7), we obtain more explicit hybrid equations, which can be written in the state form

$$m_s \dot{\boldsymbol{V}}_s + \tilde{\boldsymbol{S}}_s^T \dot{\boldsymbol{\omega}}_s + \int_{\mathcal{D}_s} \rho_s \dot{\boldsymbol{v}}_s \, \mathrm{d}\mathcal{D}_s = (2\tilde{\boldsymbol{S}}_{vs} + m_s \tilde{\boldsymbol{V}}_s + \tilde{\boldsymbol{\omega}}_s \tilde{\boldsymbol{S}}_s)\boldsymbol{\omega}_s + C_s\left(\frac{\partial L_s}{\partial \boldsymbol{R}_s}\right) + \boldsymbol{F}_s \tag{9a}$$

$$\tilde{\boldsymbol{S}}_s \dot{\boldsymbol{V}}_s + \boldsymbol{J}_s \dot{\boldsymbol{\omega}}_s + \int_{\mathcal{D}_s} \rho_s(\tilde{\boldsymbol{r}}_s + \tilde{\boldsymbol{u}}_s)\dot{\boldsymbol{v}}_s \, \mathrm{d}\mathcal{D}_s = (\tilde{\boldsymbol{S}}_s \tilde{\boldsymbol{V}}_s - \tilde{\boldsymbol{\omega}}_s \boldsymbol{J}_s - \boldsymbol{J}_{vs})\boldsymbol{\omega}_s - \tilde{\boldsymbol{\omega}}_s \int_{\mathcal{D}_s} \rho_s(\tilde{\boldsymbol{r}}_s + \tilde{\boldsymbol{u}}_s)\boldsymbol{v}_s \, \mathrm{d}\mathcal{D}_s + (D_s^T)^{-1}\left(\frac{\partial L_s}{\partial \boldsymbol{\theta}_s}\right) + \boldsymbol{M}_s \tag{9b}$$

$$\rho_s\left[\dot{\boldsymbol{V}}_s + (\tilde{\boldsymbol{r}}_s + \tilde{\boldsymbol{u}}_s)^T \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{v}}_s\right] = \rho_s(\tilde{\boldsymbol{V}}_s + 2\tilde{\boldsymbol{v}}_s)\boldsymbol{\omega}_s - \rho_s \tilde{\boldsymbol{\omega}}_s^2(\boldsymbol{r}_s + \boldsymbol{u}_s) - \mathcal{L}_s \boldsymbol{u}_s + \hat{\boldsymbol{U}}_s \tag{9c}$$

where

$$\tilde{\boldsymbol{S}}_{vs} = \tilde{\boldsymbol{S}}_s = \int_{\mathcal{D}_s} \rho_s \tilde{\boldsymbol{v}}_s \, \mathrm{d}\mathcal{D}_s \tag{10a}$$

$$\boldsymbol{J}_{vs} = \dot{\boldsymbol{J}}_s = \int_{\mathcal{D}_s} \rho_s\left[\tilde{\boldsymbol{v}}_s(\tilde{\boldsymbol{r}}_s + \tilde{\boldsymbol{u}}_s)^T + (\tilde{\boldsymbol{r}}_s + \tilde{\boldsymbol{u}}_s)\tilde{\boldsymbol{v}}_s^T\right] \mathrm{d}\mathcal{D}_s \tag{10b}$$

The complete set of state equations is obtained by augmenting Eqs. (9) with the relations

$$\dot{\boldsymbol{R}}_s = C_s^T \boldsymbol{V}_s \tag{11a}$$

$$\dot{\boldsymbol{\theta}}_s = D_s^{-1} \boldsymbol{\omega}_s \tag{11b}$$

$$\dot{\boldsymbol{u}}_s = \boldsymbol{v}_s \tag{11c}$$

where Eqs. (11a) and (11b) are merely the inverse of Eqs. (3a) and (3b), respectively, and Eq. (11c) represents a definition introduced earlier.

III. Spatial Discretization of the Partial Differential Equations

For chains of flexible substructures such as those under consideration here, no closed-form solution of the hybrid differential equations is possible, so that we must be content with an approximate solution. This implies discretization in space of the partial differential equations, which amounts to replacing the partial differential equations by sets of ordinary differential equations. The discretization can be carried out by means of the finite element method or the classical Rayleigh-Ritz method.[12] In either case, we express the elastic displacement vector in the following form:

$$\boldsymbol{u}_s(P_s, t) = \Phi_s(P_s)\boldsymbol{q}_s(t), \qquad s = 1,2,\ldots,N \tag{12}$$

where Φ_s are matrices of admissible functions and $\boldsymbol{q}_s$ are vectors of generalized displacements. The nature of the admissible functions is discussed in Ref. 13.

Introducing Eq. (12) into Eq. (4), we obtain the kinetic energy in the discretized form:

$$T_s = \tfrac{1}{2} m_s \boldsymbol{V}_s^T \boldsymbol{V}_s + \boldsymbol{V}_s^T \tilde{\boldsymbol{S}}_{s0}^T \boldsymbol{\omega}_s + \boldsymbol{V}_s^T \bar{\Phi}_s \dot{\boldsymbol{q}}_s + \boldsymbol{V}_s^T \tilde{\boldsymbol{\omega}}_s \bar{\Phi}_s \boldsymbol{q}_s + \tfrac{1}{2}\boldsymbol{\omega}_s^T \boldsymbol{J}_{s0} \boldsymbol{\omega}_s + \boldsymbol{\omega}_s^T \tilde{\Phi}_s \dot{\boldsymbol{q}}_s + \boldsymbol{\omega}_s^T \hat{\Phi}_s \boldsymbol{q}_s + \tfrac{1}{2}\dot{\boldsymbol{q}}_s^T M_s \dot{\boldsymbol{q}}_s - \tfrac{1}{2}\boldsymbol{q}_s^T \bar{H}_s \boldsymbol{q}_s - \boldsymbol{q}_s^T \tilde{H}_s \dot{\boldsymbol{q}}_s \tag{13}$$

where

$$\tilde{\boldsymbol{S}}_{s0} = \int_{\mathcal{D}_s} \rho_s \tilde{\boldsymbol{r}}_s \, \mathrm{d}\mathcal{D}_s, \qquad \boldsymbol{J}_{s0} = \int_{\mathcal{D}_s} \rho_s \tilde{\boldsymbol{r}}_s^T \tilde{\boldsymbol{r}}_s \, \mathrm{d}\mathcal{D}_s$$

$$\bar{\Phi}_s = \int_{\mathcal{D}_s} \rho_s \Phi_s \, \mathrm{d}\mathcal{D}_s, \qquad \tilde{\Phi}_s = \int_{\mathcal{D}_s} \rho_s \tilde{\boldsymbol{r}}_s \Phi_s \, \mathrm{d}\mathcal{D}_s$$

$$\hat{\Phi}_s = \int_{\mathcal{D}_s} \rho_s \tilde{\boldsymbol{r}}_s \tilde{\boldsymbol{\omega}}_s \Phi_s \, \mathrm{d}\mathcal{D}_s, \qquad M_s = \int_{\mathcal{D}_s} \rho_s \Phi_s^T \Phi_s \, \mathrm{d}\mathcal{D}_s \tag{14}$$

$$\tilde{H}_s = \tilde{H}_s(\tilde{\boldsymbol{\omega}}_s) = \int_{\mathcal{D}_s} \rho_s \Phi_s^T \tilde{\boldsymbol{\omega}}_s \Phi_s \, \mathrm{d}\mathcal{D}_s$$

$$\bar{H}_s = \bar{H}_s(\tilde{\boldsymbol{\omega}}_s) = \int_{\mathcal{D}_s} \rho_s \Phi_s^T \tilde{\boldsymbol{\omega}}_s^2 \Phi_s \, \mathrm{d}\mathcal{D}_s$$

Moreover, inserting Eq. (12) into Eq. (6), we obtain the potential energy in the discretized form:

$$V_s = \tfrac{1}{2}\boldsymbol{q}_s^T K_s \boldsymbol{q}_s + V_{gs} \tag{15}$$

where

$$K_s = [\Phi_s, \Phi_s] \tag{16}$$

represents the substructure stiffness matrix.

Equations (7a) and (7b) retain their form. On the other hand, Eq. (7c) must be replaced by its discretized version, or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_s}{\partial \boldsymbol{p}_s}\right) - \left(\frac{\partial L_s}{\partial \boldsymbol{q}_s}\right) = \boldsymbol{Q}_s \tag{17}$$

where $\boldsymbol{p}_s = \dot{\boldsymbol{q}}_s$ and $\boldsymbol{Q}_s$ is a generalized force vector whose expression is given in the next section. If the gravitational potential energy is sufficiently small compared to the elastic potential energy so that it can be ignored, then L_s does not depend

explicitly on $\boldsymbol{R}_s$ and θ_s. It follows that the terms $\partial L_s/\partial \boldsymbol{R}_s$ and $\partial L_s/\partial \theta_s$ can be omitted from Eqs. (9a) and (9b), respectively. Then the discretized version of the state equations, Eqs. (9), can be verified to be

$$m_s\dot{V}_s + \tilde{S}_{s0}^T\dot{\omega}_s + \tilde{\Phi}_s\dot{p}_s = -m_s\tilde{\omega}_s V_s - \tilde{\omega}_s\tilde{S}_{s0}^T\omega_s - 2\tilde{\omega}_s\tilde{\Phi}_s p_s - (\tilde{\dot{\omega}}_s + \tilde{\omega}_s^2)\tilde{\Phi}_s q_s + F_s \tag{18a}$$

$$\tilde{S}_{s0}\dot{V}_s + J_{s0}\dot{\omega}_s + \tilde{\Phi}_s\dot{p}_s = -\tilde{S}_{s0}\tilde{\omega}_s V_s - \tilde{\omega}_s J_{s0}\omega_s - 2\tilde{\Phi}_s p_s - \left\{([\widetilde{\dot{V}_s\omega_s}] - \tilde{\dot{V}}_s)\Phi_s + 2\tilde{\Phi}_s + 2\tilde{\omega}_s\tilde{\Phi}_s - (\tilde{\dot{\omega}}_s + \tilde{\omega}_s^2)\tilde{\Phi}_s\right\} q_s + M_s \tag{18b}$$

$$\Phi_s^T\dot{V}_s + \tilde{\Phi}_s^T\dot{\omega}_s + M_s\dot{p}_s = -\tilde{\Phi}_s^T\tilde{\omega}_s V_s + \hat{\Phi}_s^T\omega_s - 2\tilde{H}_s p_s - \left[K_s + \bar{H}_s + \dot{\bar{H}}_s)\right] q_s + Q_s \tag{18c}$$

which must be augmented by

$$\dot{R}_s = C_s^T V_s, \qquad \dot{\theta}_s = D_s^{-1}\omega_s, \qquad \dot{q}_s = p_s \tag{19}$$

IV. Generalized Forces and Torques in Terms of Actuator Forces and Torques

The terms $\boldsymbol{F}_s$, $\boldsymbol{M}_s$, and $\boldsymbol{Q}_s$ on the right side of Eqs. (18) can be regarded as generalized force and torque vectors. Our object is to express $\boldsymbol{F}_s$, $\boldsymbol{M}_s$, and $\boldsymbol{Q}_s$ in terms of the actuator forces and torques. We propose to accomplish this by means of the virtual work expression. To this end, we use the analogy with Eq. (2), consider Eq. (3a), and write the virtual displacement vector

$$\delta W_s = C_s\delta R_s + (\tilde{r}_s + \tilde{u}_s)^T\delta\alpha_s + \delta u_s \tag{20a}$$

where $\delta\boldsymbol{\alpha}_s$ is a vector of virtual quasiangular displacements of the reference frame $x_sy_sz_s$. Moreover, the vector of virtual quasiangular displacements of the body can be expressed in the following form:

$$\delta\Theta_s = \delta\alpha_s + \nabla\times\delta u_s \tag{20b}$$

where $\nabla\times\delta\boldsymbol{u}_s$ is the contribution of the elastic motions. We observe that $\delta\boldsymbol{W}_s$ and $\delta\boldsymbol{\Theta}_s$ are in terms of body axes components. Introducing Eq. (12) into Eqs. (20), we obtain the discretized versions:

$$\delta W_s = C_s\delta R_s + (\tilde{r}_s + \widetilde{\Phi_s q_s})^T\delta\alpha_s + \Phi_s\delta q_s \tag{21a}$$

$$\delta\Theta_s = \delta\alpha_s + \Psi_s\delta q_s \tag{21b}$$

where Ψ_s is a matrix defined by

$$\nabla\times\Phi_s q_s = \Psi_s q_s \tag{22}$$

Next, we assume that the control is implemented by distributed actuators applying the force density $\boldsymbol{f}_s(r_s,t)$ and torque density $\boldsymbol{m}_s(r_s,t)$. Then, using Eqs. (21), the virtual work performed by the actuators can be expressed as

$$\begin{aligned}\delta W_s &= \int_{\mathfrak{D}_s}(f_s^T\delta W_s + m_s^T\delta\Theta_s)\,d\mathfrak{D}_s \\ &= \int_{\mathfrak{D}_s}\left[f_s^T(C_s\delta R_s + (\tilde{r}_s + \widetilde{\Phi_s q_s})^T\delta\alpha_s + \Phi_s\delta q_s) + m_s^T(\delta\alpha_s + \Psi_s\delta q_s)\right]d\mathfrak{D}_s \\ &= \int_{\mathfrak{D}_s} f_s^T\,d\mathfrak{D}_s C_s\delta R_s + \int_{\mathfrak{D}_s}\left[f_s^T(\tilde{r}_s + \widetilde{\Phi_s q_s})^T + m_s^T\right]d\mathfrak{D}_s\delta\alpha_s \\ &\quad + \int_{\mathfrak{D}_s}(f_s^T\Phi_s + m_s^T\Psi_s)\,d\mathfrak{D}_s\delta q_s\end{aligned} \tag{23}$$

On the other hand, the same virtual work can be expressed in terms of the generalized forces and torques in the following form:

$$\delta W_s = F_s^T C_s\delta R_s + M_s^T\delta\alpha_s + Q_s^T\delta q_s \tag{24}$$

so that, comparing Eqs. (23) and (24), we conclude that the desired relations are

$$F_s = \int_{\mathfrak{D}_s} f_s\,d\mathfrak{D}_s \tag{25a}$$

$$M_s = \int_{\mathfrak{D}_s}\left[(\tilde{r}_s + \widetilde{\Phi_s q_s})f_s + m_s\right]d\mathfrak{D}_s \tag{25b}$$

$$Q_s = \int_{\mathfrak{D}_s}(\Phi_s^T f_s + \Psi_s^T m_s)\,d\mathfrak{D}_s \tag{25c}$$

Distributed actuators are not within the state of the art, so that control implementation must be carried out by discrete actuators, such as point actuators. But point actuators can be treated as distributed by writing

$$f_s = \sum_{i=1}^{n_s} f_{si}(t)\delta(r_s - r_{si}) \tag{26a}$$

$$m_s = \sum_{j=1}^{m_s} m_{sj}(t)\delta(r_s - r_{sj}) \tag{26b}$$

where $\boldsymbol{f}_{si}$ and $\boldsymbol{m}_{sj}$ are time-dependent amplitudes and $\delta(r_s - r_{si})$ and $\delta(r_s - r_{sj})$ are spatial Dirac delta functions defined by

$$\delta(r_s - r_{sk}) = 0, \qquad r_s \neq r_{sk}, \qquad k = i,j \tag{27a}$$

$$\int_{\mathfrak{D}_s}\delta(r_s - r_{sk})\,d\mathfrak{D}_s = 1, \qquad k = i,j \tag{27b}$$

where r_{sk} denotes actuator locations. Hence, inserting Eqs. (26) into Eqs. (25) and considering Eqs. (27), we obtain

$$F_s = \sum_{i=1}^{n_s} f_{si} \tag{28a}$$

$$M_s = \sum_{i=1}^{n_s}\left[\tilde{r}_{si} + \widetilde{\Phi(r_{si})q_s}\right]f_{si} + \sum_{j=1}^{m_s} m_{sj} \tag{28b}$$

$$Q_s = \sum_{i=1}^{n_s}\Phi_s^T(r_{si})\,f_{si} + \sum_{j=1}^{m_s}\Psi_s^T(r_{sj})m_{sj} \tag{28c}$$

The equations of motion in terms of actuator forces and torques are obtained by introducing Eqs. (28) into Eqs. (18).

V. Perturbation Approach to the Maneuvering and Control Design

Designing the maneuvering and control for articulated systems of substructures is very difficult, especially if the design is to be optimal in some fashion. The difficulty can be traced to the fact that the system is nonlinear and of high order. The nonlinearity can be attributed to the rigid-body motions and the high order to the elastic motions. The perturbation approach is based on the simple observation that rigid-body motions tend to be large compared to the elastic motions. Consistent with this, we introduce the following notation:

$$\begin{aligned} V_s &= V_{s0} + V_{s1}, & \omega_s &= \omega_{s0} + \omega_{s1} \\ f_{si} &= f_{si0} + f_{si1}, & m_{sj} &= m_{sj0} + m_{sj1} \end{aligned} \tag{29}$$

where the subscript 0 denotes zero-order quantities and the subscript 1 denotes first-order quantities, in which zero order implies one order of magnitude larger than first order. The elastic motions are by definition first order in magnitude.

Inserting Eqs. (29) into Eqs. (18a) and (18b), recalling Eqs. (28a) and (28b) and separating order of magnitude, we obtain the zero-order state equations characterizing the maneuver

$$m_s\dot{V}_{s0} + \tilde{S}_{s0}^T\dot{\omega}_{s0} = -m_s\tilde{\omega}_{s0}V_{s0} + \tilde{\omega}_{s0}\tilde{S}_{s0}\omega_{s0} + \sum_{i=1}^{n_s} f_{si0} \tag{30a}$$

$$\tilde{S}_{s0}\dot{V}_{s0} + J_{s0}\dot{\omega}_{s0} = -\tilde{S}_{s0}\tilde{\omega}_{s0}V_{s0} - \tilde{\omega}_{s0}J_{s0}\omega_{s0} + \sum_{i=1}^{n_s}\tilde{r}_{si}f_{si0} + \sum_{j=1}^{m_s} m_{sj0} \tag{30b}$$

Moreover, the first two equations in (19) yield

$$\dot{R}_{s0} = C_{s0}^T V_{s0}, \qquad \dot{\theta}_{s0} = D_{s0}^{-1}\omega_{s0} \tag{31}$$

where $C_{s0}^T = C_s^T(\theta_{s0})$ and $D_{s0}^{-1} = D_s^{-1}(\theta_{s0})$. Equations (30) and (31) constitute the state equations for the zero-order problem and can be cast in a compact form. To this end, we introduce the substructure state vector

$$x_{s0}(t) = \left[R_{s0}^T(t)\ \theta_{s0}^T(t)\ V_{s0}^T(t)\ \omega_{s0}^T(t)\right]^T \tag{32a}$$

where θ_{s0} is a vector only in a symbolic way, and the substructure actuator force vector

$$f_{s0}(t) = \left[f_{s10}^T(t)\ f_{s20}^T(t)\cdots f_{sn_s0}^T(t)\ m_{s10}^T(t)\ m_{s20}^T(t)\cdots m_{sm_s0}^T(t)\right]^T \tag{32b}$$

Then, Eqs. (30) and (31) can be rewritten in matrix form

$$\mathfrak{M}_{s0}\dot{x}_{s0} = \mathfrak{C}_{s0}x_{s0} + \mathfrak{B}_{s0}f_{s0}, \qquad s = 1,2,\ldots,N \tag{33}$$

where

$$\mathfrak{M}_{s0} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & m_sI & \tilde{S}_{s0}^T \\ 0 & 0 & \tilde{S}_{s0} & J_{s0} \end{bmatrix} \tag{34a}$$

$$\mathfrak{C}_{s0} = \begin{bmatrix} 0 & 0 & C_{s0}^T & 0 \\ 0 & 0 & 0 & D_{s0}^{-1} \\ 0 & 0 & -m_s\tilde{\omega}_{s0} & \tilde{\omega}_{s0}\tilde{S}_{s0} \\ 0 & 0 & -\tilde{S}_{s0}\tilde{\omega}_{s0} & -\tilde{\omega}_{s0}J_{s0} \end{bmatrix} \tag{34b}$$

$$\mathfrak{B}_{s0} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ I & I & \cdots & I & 0 & 0 & \cdots & 0 \\ \tilde{r}_{s1} & \tilde{r}_{s2} & \cdots & \tilde{r}_{sn_s} & I & I & \cdots & I \end{bmatrix} \tag{34c}$$

in which $\mathfrak{M}_{s0}$ and $\mathfrak{C}_{s0}$ are 12×12 matrices and $\mathfrak{B}_{s0}$ is a $12\times 3(n_s+m_s)$ matrix.

The preceding process also yields the first-order equations

$$m_s\dot{V}_{s1} + \tilde{S}_{s0}^T\dot{\omega}_{s1} + \tilde{\Phi}_s\dot{p}_s = -m_s\tilde{\omega}_{s0}V_{s1} - \Gamma_s\omega_{s1} - 2\tilde{\omega}_{s0}\tilde{\Phi}_s p_s - \aleph_s q_s + \sum_{i=1}^{n_s} f_{si1} \tag{35a}$$

$$\tilde{S}_{s0}\dot{V}_{s1} + J_{s0}\dot{\omega}_{s1} + \tilde{\Phi}_s\dot{p}_s = -\tilde{S}_{s0}\tilde{\omega}_{s0}V_{s1} - \Delta_s\omega_{s1} - 2\hat{\Phi}_s(\omega_{s0})p_s - \Xi_s q_s + \sum_{i=1}^{n_s}\tilde{r}_{si}f_{si1} + \sum_{j=1}^{m_s} m_{sj1} \tag{35b}$$

$$\tilde{\Phi}_s^T\dot{V}_{s1} + \tilde{\Phi}_s^T\dot{\omega}_{s1} + M_s\dot{p}_s = -\tilde{\Phi}_s^T\tilde{\omega}_{s0}V_{s1} - \Upsilon_s\omega_{s1} - 2\tilde{H}_s(\omega_{s0})p_s - \bar{K}_s q_s + \sum_{i=1}^{n_s}\Phi_s^T(r_{si})f_{si1} + \sum_{j=1}^{m_s}\Psi_s^T(r_{sj})m_{sj1} + \delta_{s1} \tag{35c}$$

where

$$\begin{aligned} \Gamma_s &= \tilde{S}_{s0}\tilde{\omega}_{s0} - 2\tilde{\omega}_{s0}\tilde{S}_{s0} - m_s\tilde{V}_{s0} \\ \aleph_s &= (\tilde{\omega}_{s0} + \tilde{\omega}_{s0}^2)\tilde{\Phi}_{s0} + \sum_{i=1}^{n_s}\tilde{f}_{si0}\Phi_s(r_{si}) \\ \Delta_s &= 2\tilde{\omega}_{s0}J_{s0} + J_{s0}\tilde{\omega}_{s0} - (\operatorname{tr} J_{s0})\tilde{\omega}_{s0} - \tilde{S}_{s0}\tilde{V}_{s0} \\ \Xi_s &= \left(\widetilde{[\tilde{V}_{s0}\omega_{s0}]} - \tilde{V}_{s0}\right)\tilde{\Phi}_s + 2\hat{\Phi}_s(\dot{\omega}_{s0}) + 2\tilde{\omega}_{s0}\hat{\Phi}_s(\omega_{s0}) - (\tilde{\omega}_{s0} + \tilde{\omega}_{s0}^2)\tilde{\Phi}_s \\ \Upsilon_s &= -\tilde{\Phi}_s^T\tilde{\omega}_{s0} - 2\hat{\Phi}_s^T(\omega_{s0}) - \tilde{\Phi}_s^T\tilde{V}_{s0} \\ \bar{K}_s &= K_s + \bar{H}_s(\omega_{s0}) + \tilde{H}_s(\dot{\omega}_{s0}) \\ \delta_{s1} &= -\tilde{\Phi}_s^T(\dot{V}_{s0} + \tilde{\omega}_{s0}V_{s0}) - \tilde{\Phi}_s\dot{\omega}_{s0} + \hat{\Phi}_s^T(\omega_{s0})\omega_{s0} + \sum_{i=1}^{n_s}\Phi_s^T(r_{si})f_{si0} + \sum_{j=1}^{m_s}\Psi_s^T(r_{sj})m_{sj0} \end{aligned} \tag{36}$$

in which tr denotes the trace of the matrix.

The state equations can be obtained by augmenting Eqs. (35) with suitable kinematical relations between body axes and inertial perturbations in the translational and angular displacements. Denoting by U_{s1} and β_{s1} the body-axes vectors of perturbations in the translational and angular displacements, respectively, we can write these relations as follows:

$$U_{s1} = C_{s0}R_{s1}, \qquad \beta_{s1} = D_{s0}\theta_{s1} \tag{37}$$

It can be verified that the time derivatives of the body-axes perturbation vectors have the expressions

$$\dot{U}_{s1} = -\tilde{\omega}_{s0}U_{s1} - \tilde{V}_{s0}\beta_{s1} + V_{s1} \tag{38a}$$

$$\dot{\beta}_{s1} = -\tilde{\omega}_{s0}\beta_{s1} + \omega_{s1} \tag{38b}$$

Then, introducing the perturbation state vector

$$x_{s1}(t) = \left[U_{s1}^T(t)\ \beta_{s1}^T(t)\ q_s^T(t)\ V_{s1}^T(t)\ \omega_{s1}^T(t)\ p_s^T(t)\right]^T \tag{39a}$$

and the perturbation force vector

$$f_{s1}(t) = \left[f_{s11}^T(t)\ f_{s21}^T(t)\cdots f_{sn_s1}^T(t)\ m_{s11}^T(t)\ m_{s21}^T(t)\cdots m_{sm_s1}^T(t)\right]^T \tag{39b}$$

and recalling the last equation in (19), the first-order perturbation state equations can be written in matrix form

$$\mathfrak{M}_{s1}\dot{x}_{s1} = \mathfrak{C}_{s1}x_{s1} + \mathfrak{B}_{s1}f_{s1} + d_{s1}, \qquad s = 1,2,\ldots,N \tag{40}$$

where

$$\mathfrak{M}_{s1} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & m_sI & \tilde{S}_{s0}^T & \tilde{\Phi}_s \\ 0 & 0 & 0 & \tilde{S}_{s0} & J_{s0} & \tilde{\Phi}_s \\ 0 & 0 & 0 & \tilde{\Phi}_s^T & \tilde{\Phi}_s^T & M_s \end{bmatrix} \tag{41a}$$

$$\mathcal{C}_{s1} = \begin{bmatrix} -\tilde{\omega}_{s0} & -\tilde{V}_{s0} & 0 & I & 0 & 0 \\ 0 & -\tilde{\omega}_{s0} & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & -\aleph_s & -m_s\tilde{\omega}_{s0} & -\Gamma_s & -2\tilde{\omega}_{s0}\bar{\Phi}_s \\ 0 & 0 & -\Xi_s & -\tilde{S}_s\tilde{\omega}_{s0} & -\Delta_s & -2\hat{\Phi}_s(\omega_{s0}) \\ 0 & 0 & -\bar{K}_s & -\bar{\Phi}_s^T\tilde{\omega}_{s0} & -\Upsilon_s & -2\tilde{H}_s(\omega_{s0}) \end{bmatrix} \tag{41b}$$

$$\mathcal{B}_{s1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ I & I & \cdots & I & 0 & 0 & \cdots & 0 \\ \tilde{r}_{s1} & \tilde{r}_{s2} & \cdots & \tilde{r}_{sn_s} & I & I & \cdots & I \\ \Phi_s^T(r_{s1}) & \Phi_s^T(r_{s2}) & \cdots & \Phi_s^T(r_{sn_s}) & \Psi_s^T(r_{s1}) & \Psi_s^T(r_{s2}) & \cdots & \Psi_s^T(r_{sm_s}) \end{bmatrix} \tag{41c}$$

$$d_{s1} = \left[\mathbf{0}^T \ \mathbf{0}^T \ \mathbf{0}^T \ \mathbf{0}^T \ \mathbf{0}^T \ \delta_{s1}^T\right]^T \tag{41d}$$

VI. Kinematical Synthesis for the Zero-Order Equations

The derivation of the equations for the individual substructures was carried out in Sec. II as if the substructures were free to move relative to one another. In reality, however, they are all part of an interacting system of flexible bodies, so that the motions of the various substructures are coupled. Indeed, because the various substructures are hinged to one another, the motion of a given hinge is accounted for several times, once for each substructure sharing the hinge. As a result, the motions defined by the independent equations of motion derived in Sec. II contain redundant coordinates. In this regard, we note that hinges constrain the translational motions but leave the rotational motions free. The kinematical synthesis to be introduced soon is designed to eliminate the surplus coordinates. To this end, we consider two adjacent substructures in the chain, say, p and q, as shown in Fig. 2. We propose to carry out the kinematical synthesis for the zero-order and first-order equations separately. Then, with reference to Fig. 2, we conclude that the zero-order rigid-body displacements and velocities of the two substructures are related as follows:

$$\boldsymbol{R}_{q0} = \boldsymbol{R}_{p0} + C_{p0}^T\boldsymbol{L}_{pq} \tag{42a}$$

$$\boldsymbol{\theta}_{q0} = \boldsymbol{\theta}_{q0} \tag{42b}$$

$$\boldsymbol{V}_{q0} = C_{qp}(\boldsymbol{V}_{p0} - \tilde{L}_{pq}\boldsymbol{\omega}_{p0}) \tag{42c}$$

$$\boldsymbol{\omega}_{q0} = \boldsymbol{\omega}_{q0} \tag{42d}$$

where $\boldsymbol{L}_{pq}$ is a radius vector from p to q and $C_{qp} = C_{q0}C_{p0}^T$ is the matrix of direction cosines between $x_qy_qz_q$ and $x_py_pz_p$. From Eqs. (42a) and (42c), we conclude that the position vector and translational velocity vector of one substructure are independent. We choose arbitrarily this substructure to be the first, $s = 1$. Repeated use of Eqs. (42a) and (42c) permits us to write the following recursive formula for the position and velocity vectors of substructure s:

$$\boldsymbol{R}_{s0} = \boldsymbol{R}_{10} + \sum_{j=1}^{s-1} C_{j,0}^T\boldsymbol{L}_{j,j+1} \tag{43a}$$

$$\boldsymbol{V}_{s0} = C_{s1}\boldsymbol{V}_{10} - \sum_{j=1}^{s-1} C_{sj}\tilde{L}_{j,j+1}\boldsymbol{\omega}_{j0} \tag{43b}$$

Equations (43) represent constraint equations, which can be used to eliminate the surplus coordinates implicit in Eqs. (33). Indeed, using Eqs. (42b), (42d), and (43), we obtain a recursive formula relating the state of substructure s to the position of the independent state corresponding to the aggregation of substructures 1 through s in the following form:

$$\boldsymbol{x}_{s0}^u = T_{s0}\boldsymbol{x}_{s0}^c + \boldsymbol{e}_{s0}, \qquad s = 1,2,\ldots,N \tag{44}$$

where $\boldsymbol{x}_{s0}^u = \boldsymbol{x}_{s0}$ is the 12-dimensional substructure state vector defined by Eq. (32a),

$$\boldsymbol{x}_{s0}^c = \left[\boldsymbol{R}_{10}^T \ \boldsymbol{\theta}_{10}^T \ \boldsymbol{\theta}_{20}^T \cdots \boldsymbol{\theta}_{s0}^T \ \boldsymbol{V}_{10}^T \ \boldsymbol{\omega}_{10}^T \ \boldsymbol{\omega}_{20}^T \cdots \boldsymbol{\omega}_{s0}^T\right]^T \tag{45a}$$

$$\boldsymbol{e}_{s0} = \begin{bmatrix} \sum_{j=1}^{s-1} C_{j,0}^T\boldsymbol{L}_{j,j+1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{45b}$$

are a partial $6(s+1)$-dimensional state vector and a partial 12-dimensional position vector, respectively, and

$$T_{s0} = \left[T_{s0}^{(1)} \ \ T_{s0}^{(2)}\right] \tag{46}$$

is a $12 \times 6(s+1)$ matrix, in which

$$T_{s0}^{(1)} = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{47a}$$

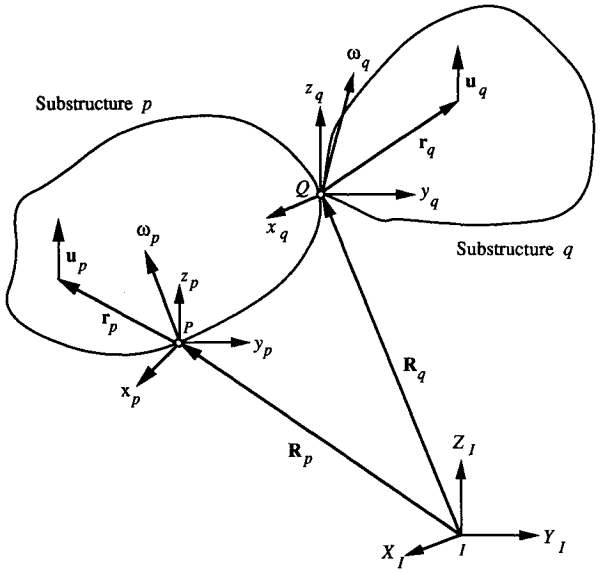

Fig. 2 Two adjacent substructures in the chain.

$$T_{s0}^{(2)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ C_{s1} & -C_{s1}\tilde{L}_{12} & \cdots & -C_{s,s-1}\tilde{L}_{s-1,s} & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{47b}$$

are $12\times 3(s+1)$ submatrices. Note that the superscripts u and c indicate unconstrained and constrained vectors, respectively, where the latter represents a vector of independent state variables.

Equations (33) represent a set of disjoint equations, which implies that the equations for one substructure are independent of those for another substructure. To obtain a set of equations describing the motion of our system of interacting substructures, the equations must be transformed into a set in terms of entirely independent state variables. To this end, we first combine Eqs. (33) into

$$\mathfrak{M}_0\dot{\boldsymbol{x}}_0^u = \mathfrak{C}_0\boldsymbol{x}_0^u + \mathfrak{B}_0\boldsymbol{f}_0 \tag{48}$$

where

$$\boldsymbol{x}_0^u = \begin{bmatrix} \boldsymbol{x}_{10} \\ \boldsymbol{x}_{20} \\ \vdots \\ \boldsymbol{x}_{N0} \end{bmatrix}, \qquad \boldsymbol{f}_0 = \begin{bmatrix} \boldsymbol{f}_{10} \\ \boldsymbol{f}_{20} \\ \vdots \\ \boldsymbol{f}_{N0} \end{bmatrix} \tag{49}$$

are $12N$-dimensional and $3\Sigma_{s=1}^{N}(n_s+m_s)$-dimensional vectors, respectively, and

$$\begin{aligned} \mathfrak{M}_0 &= \text{block diag}(\mathfrak{M}_{s0}) \\ \mathfrak{C}_0 &= \text{block diag}(\mathfrak{C}_{s0}) \\ \mathfrak{B}_0 &= \text{block diag}(\mathfrak{B}_{s0}) \end{aligned} \tag{50}$$

are $12N\times 12N$, $12N\times 12N$ and $12N\times 3\Sigma_{s=1}^{N}(n_s+m_s)$ matrices of coefficients, respectively. To achieve our goal, we must invoke Eqs. (44). The process can be carried out conveniently by introducing the $12N$-dimensional vectors

$$\boldsymbol{x}_{10}^* = \begin{bmatrix} \boldsymbol{x}_{10} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{x}_{20}^* = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{x}_{20} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}, \quad \ldots, \quad \boldsymbol{x}_{N0}^* = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{x}_{N0} \end{bmatrix} \tag{51}$$

Then we can write

$$\boldsymbol{x}_0^u = \sum_{s=1}^{N} \boldsymbol{x}_{s0}^* \tag{52}$$

and

$$T_{10}^* = \begin{bmatrix} T_{10}^{(1)} & 0 & \cdots & 0 & T_{10}^{(2)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$T_{20}^* \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ T_{20}^{(1)} & \rightarrow & \cdots & 0 & T_{20}^{(2)} & \rightarrow & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ldots, T_{N0}^* = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \leftarrow & & T_{N0} & & \rightarrow \end{bmatrix} \tag{53}$$

where $T_{10}^{(1)}$ and $T_{10}^{(2)}$ are 12×6 matrices, $T_{20}^{(1)}$ and $T_{20}^{(2)}$ are 12×9 matrices, and T_{N0} is a $12\times 6(N+1)$ matrix, as well as the $12N$-dimensional vector

$$\boldsymbol{e}_0 = [\boldsymbol{e}_{10}^T \quad \boldsymbol{e}_{20}^T \quad \cdots \quad \boldsymbol{e}_{N0}^T] \tag{54}$$

This permits us to combine the constraint equations, Eqs. (44), into

$$\boldsymbol{x}_0^u = \sum_{s=1}^{N} \boldsymbol{x}_{s0}^* = T_0\boldsymbol{x}_0^c + \boldsymbol{e}_0 \tag{55}$$

where, with reference to Eq. (45a), $\boldsymbol{x}_0^c = \boldsymbol{x}_{N0}^c$. Moreover, $T_0 = \Sigma_{s=1}^{N} T_{s0}^*$.

Inserting Eq. (55) into Eq. (48) and rearranging, we have

$$\mathfrak{M}_0 T_0\dot{\boldsymbol{x}}_0^c = (\mathfrak{C}_0 T_0 - \mathfrak{M}_0\dot{T}_0)\boldsymbol{x}_0^c + \mathfrak{B}_0\boldsymbol{f}_0 + \mathfrak{C}_0\boldsymbol{e}_0 - \mathfrak{M}_0\dot{\boldsymbol{e}}_0 \tag{56}$$

Finally, multiplying Eq. (56) on the left by T_0^T and then multiplying the resulting equation on the left by $(T_0^T\mathfrak{M}_0T_0)^{-1}$, we obtain the state equation for the zero-order problem in the following form:

$$\dot{\boldsymbol{x}}_0 = \mathfrak{A}_0\boldsymbol{x}_0 + \mathfrak{B}_0^*\boldsymbol{f}_0 + \boldsymbol{d}_0^* \tag{57}$$

where

$$\mathfrak{A}_0 = (T_0^T\mathfrak{M}_0T_0)^{-1}T_0^T(\mathfrak{C}_0T_0 - \mathfrak{M}_0\dot{T}_0) \tag{58a}$$

$$\mathfrak{B}_0^* = (T_0^T\mathfrak{M}_0T_0)^{-1}T_0^T\mathfrak{B}_0 \tag{58b}$$

$$\boldsymbol{d}_0^* = (T_0^T\mathfrak{M}_0T_0)^{-1}T_0^T(\mathfrak{C}_{s0}\boldsymbol{e}_0 - \mathfrak{M}_0\dot{\boldsymbol{e}}_0) \tag{58c}$$

and note that we dropped the superscript c for simplicity of notation.

VII. Kinematical Synthesis for the First-Order Equations

The kinematical synthesis for the first-order equations is carried out in the same way as that for the zero-order equations. The derivation of the constraint relations is tedious and is omitted here for brevity. The basic recursive relations are

$$\begin{aligned} \boldsymbol{U}_{q1} &= C_{qp}(\boldsymbol{U}_{p1} - \tilde{L}_{pq}\boldsymbol{\beta}_{p1} + \Phi_{pq}\boldsymbol{q}_p) \\ \boldsymbol{\beta}_{q1} &= C_{qp}(\boldsymbol{\beta}_{p1} + \Psi_{pq}\boldsymbol{q}_p), \qquad \boldsymbol{q}_q = \boldsymbol{q}_q \\ \boldsymbol{V}_{q1} &= \left[\tilde{V}_{p0}C_{qp} + C_{qp}\left(\widetilde{[\tilde{L}_{pq}\boldsymbol{\omega}_{p0}]} - \tilde{V}_{p0}\right)\right]\boldsymbol{\beta}_{p1} \\ &\quad + (\tilde{V}_{p0}C_{qp}\Psi_{pq} + C_{qp}\tilde{\omega}_{p0}\Phi_{pq})\boldsymbol{q}_p \\ &\quad + C_{qp}\left[\boldsymbol{V}_{p1} - \tilde{L}_{pq}\boldsymbol{\omega}_{p1} + \Phi_{pq}\boldsymbol{p}_p\right] \\ \boldsymbol{\omega}_{q1} &= C_{qp}(\tilde{\omega}_{p0}\Psi_{pq}\boldsymbol{q}_p + \boldsymbol{\omega}_{p1} + \Psi_{pq}\boldsymbol{p}_p), \qquad \boldsymbol{p}_q = \boldsymbol{p}_q \end{aligned} \tag{59}$$

where

$$\Phi_{pq} = \Phi_p(L_{pq}), \qquad \Psi_{pq} = \Psi_p(L_{pq}) \tag{60}$$

Equations (59) can be written in compact form:

$$x_{q1} = T^*_{qp} x^*_{p1} \tag{61}$$

where

$$x_{q1} = \left[U^T_{q1} \;\; \beta^T_{q1} \;\; q^T_q \;\; V^T_{q1} \;\; \omega^T_{q1} \;\; p^T_q\right]^T \tag{62a}$$

$$x^*_{p1} = \left[U^T_{p1} \;\; \beta^T_{p1} \;\; q^T_p \;\; q^T_q \;\; V^T_{p1} \;\; \omega^T_{p1} \;\; p^T_p \;\; p^T_q\right]^T \tag{62b}$$

are substructure perturbation state vectors and

$$T^*_{qp} = \begin{bmatrix} T^{(11)}_{qp} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ T^{(21)}_{qp} & 0 & T^{(22)}_{qp} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{63}$$

is a transformation matrix, in which

$$T^{(11)}_{qp} = \begin{bmatrix} C_{qp} & -C_{qp}\tilde{L}_{pq} & C_{qp}\Phi_{pq} \\ 0 & C_{qp} & C_{qp}\Psi_{pq} \end{bmatrix} = T^{(22)}_{qp} \tag{64a}$$

$$T^{(21)}_{qp} = \begin{bmatrix} 0 & \tau^{12}_{qp} & \tau^{13}_{qp} \\ 0 & 0 & C_{qp}\tilde{\omega}_{p0}\Psi_{pq} \end{bmatrix} \tag{64b}$$

where

$$\tau^{12}_{qp} = \tilde{V}_{p0}C_{qp} + C_{qp}\left(\widetilde{[\tilde{L}_{pq}\omega_{p0}]} - \tilde{V}_{p0}\right) \tag{65a}$$

$$\tau^{13}_{qp} = \tilde{V}_{p0}C_{qp}\Psi_{pq} + C_{qp}\tilde{\omega}_{p0}\Phi_{pq} \tag{65b}$$

Considering Eq. (61), we can write

$$x^u_{s1} = T_{s1}x^c_{s1}, \qquad s = 2,3,\ldots,N \tag{66}$$

where

$$x^u_{s1} = x_{s1} \tag{67a}$$

$$T_{s1} = T_{s,s-1}\cdots T_{32}T_{21} = \prod_{j=s}^{2} T_{j,j-1} \tag{67b}$$

$$x^c_{s1} = \left[U^T_{11} \;\; \beta^T_{11} \;\; q^T_1 \;\; q^T_2 \cdots q^T_s \;\; V^T_{11} \;\; \omega^T_{11} \;\; p^T_1 \;\; p^T_2 \cdots p^T_N\right]^T \tag{67c}$$

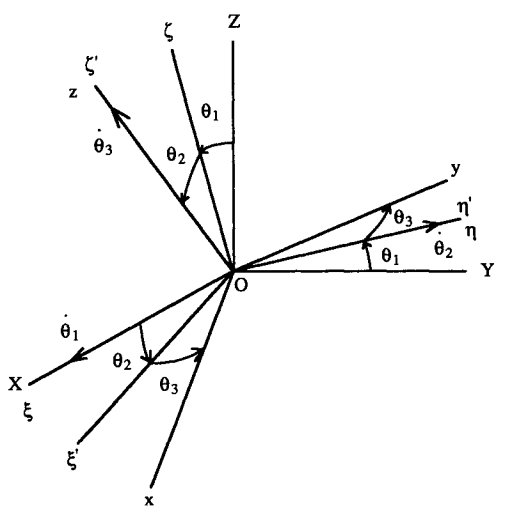

Fig. 3 Rotations leading from inertial axes to body axes.

in which $T_{j,j-1}$ can be shown to have the following form:

$$T_{j,j-1} = \begin{bmatrix} T^{(11)}_{j,j-1} & 0 & 0 & 0 \\ 0 & I_{s-j+1} & 0 & 0 \\ T^{(21)}_{j,j-1} & 0 & T^{(22)}_{j,j-1} & 0 \\ 0 & 0 & 0 & I_{s-j+1} \end{bmatrix} \tag{68}$$

As in Sec. VI, we can combine the disjoint perturbation equations into

$$\mathfrak{M}_1\dot{x}^u_1 = \mathfrak{C}_1 x^u_1 + \mathfrak{B}_1 f_1 + d_1 \tag{69}$$

where

$$x^u_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{N1} \end{bmatrix}, \qquad f_1 = \begin{bmatrix} f_{11} \\ f_{21} \\ \vdots \\ f_{N1} \end{bmatrix}, \qquad d_1 = \begin{bmatrix} d_{11} \\ d_{21} \\ \vdots \\ d_{N1} \end{bmatrix} \tag{70}$$

and

$$\begin{aligned} \mathfrak{M}_1 &= \text{block diag}(\mathfrak{M}_{s1}) \\ \mathfrak{C}_1 &= \text{block diag}(\mathfrak{C}_{s1}) \\ \mathfrak{B}_1 &= \text{block diag}(\mathfrak{B}_{s1}) \end{aligned} \tag{71}$$

Then, completing the substructure state vectors so as to read

$$x^*_{11} = \begin{bmatrix} x_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; x^*_{21} = \begin{bmatrix} 0 \\ x_{21} \\ \vdots \\ 0 \end{bmatrix}, \ldots, \; x^*_{N1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_{N1} \end{bmatrix} \tag{72}$$

and introducing the constrained perturbation state vector

$$x^c_1 = \left[U^T_{11} \;\; \beta^T_{11} \;\; q^T_1 \;\; q^T_2 \cdots q^T_N \;\; V^T_{11} \;\; \omega^T_{11} \;\; p^T_1 \;\; p^T_2 \cdots p^T_N\right]^T \tag{73}$$

we can write the relation between the unconstrained substructure vectors and the constrained vector in the following form:

$$x^*_{s1} = T^*_{s1}x^c_1, \qquad s = 2,3,\ldots,N \tag{74}$$

where T^*_{s1} are obtained from T_{s1} by adding null matrices, in a process similar to that described by Eqs. (53). Then, using Eqs. (70), the unconstrained state vector can be written as

$$x^u_1 = \sum_{s=1}^{N} x^*_{s1} = \sum_{s=1}^{N} T^*_{s1}x^c_{s1} = T_1 x^c_1 \tag{75}$$

where

$$T_1 = \sum_{s=1}^{N} T^*_{s1} \tag{76}$$

Finally, introducing Eq. (75) into Eq. (69), we obtain the state equations for the first-order perturbations

$$\mathfrak{M}_1 T_1 \dot{x}^c_1 = (\mathfrak{C}_1 T_1 - \mathfrak{M}_1 \dot{T}_1)x^c_1 + \mathfrak{B}_1 f_1 + d_1 \tag{77}$$

Multiplying Eq. (77) on the left by T_1^T and then multiplying the result on the left by $(T_1^T\mathfrak{M}_1 T_1)^{-1}$, we obtain the state equations for the zero-order problem in the following form:

$$\dot{x}_1 = \mathfrak{A}_1 x_1 + \mathfrak{B}^*_1 f_1 + d^*_1 \tag{78}$$

where

$$\mathcal{C}_1 = (T_1^T \mathfrak{M}_1 T_1)^{-1} T_1^T (\mathcal{C}_1 T_1 - \mathfrak{M}_1 \dot{T}_1)$$
$$\mathcal{B}_1^* = (T_1^T \mathfrak{M}_1 T_1)^{-1} T_1^T \mathcal{B}_1 \tag{79}$$
$$\boldsymbol{d}_1^* = (T_1^T \mathfrak{M}_1 T_1)^{-1} T_1^T \boldsymbol{d}_1$$

and we note that once again we dropped the superscript c.

VIII. Numerical Example

A computer program based on the algorithm developed here was written using FORTRAN. The program, referred to as MKLM, can handle three-dimensional problems and can accommodate systems with branches. For the set of angles θ_i $(i = 1,2,3)$ shown in Fig. 3, the matrix of direction cosines and the transformation matrix for the substructure s, defined by Eqs. (19a) and (19b), respectively, are as follows:

$$C_s = \begin{bmatrix} c\theta_{s2}c\theta_{s3} & c\theta_{s1}s\theta_{s3} + s\theta_{s1}s\theta_{s2}c\theta_{s3} & s\theta_{s1}s\theta_{s3} - c\theta_{s1}s\theta_{s2}c\theta_{s3} \\ -c\theta_{s2}s\theta_{s3} & c\theta_{s1}c\theta_{s3} - s\theta_{s1}s\theta_{s2}s\theta_{s3} & s\theta_{s1}c\theta_{s3} + c\theta_{s1}s\theta_{s2}s\theta_{s3} \\ s\theta_{s2} & -s\theta_{s1}c\theta_{s2} & c\theta_{s1}c\theta_{s2} \end{bmatrix} \tag{80}$$

$$D_s = \begin{bmatrix} c\theta_{s2}c\theta_{s3} & s\theta_{s3} & 0 \\ -c\theta_{s2}s\theta_{s3} & c\theta_{s3} & 0 \\ s\theta_{s2} & 0 & 1 \end{bmatrix} \tag{81}$$

where $c\theta_{si} = \cos\theta_{si}$ and $s\theta_{si} = \sin\theta_{si}$ $(i = 1,2,3)$.

As an example, we consider the planar motion of the three-beam system shown in Fig. 4. Each body represents a uniform slender beam of circular cross section. The maneuver consists of a 41-deg counterclockwise rotation of the center beam and is carried out according to the bang-bang control law by a torquer located at the center of the center beam. In addition, each beam is equipped with two actuators in charge of suppressing elastic vibrations and perturbations in the "rigid-body" maneuver motions. The location of the actuators is shown in Fig. 4. Three sets of body axes are considered: one with the origin at the center of beam 1 and the other with the origin at the end of beams 2 and 3. The beams undergo bending deformation in the y_s direction alone, so that the vector $\boldsymbol{u}_s$ has only one nonzero component. Consistent with this, the matrix Φ_s in Eq. (12) reduces to

$$\Phi_s(x_s) = \text{block-diag}\left[\mathbf{0}^T \quad \boldsymbol{\phi}_s^T(x_s) \quad \mathbf{0}^T\right] \tag{82}$$

It will prove convenient to introduce the nondimensional variable $\xi_s = x_s/l_s$, so that $\boldsymbol{r}_s = [x_s \ 0 \ 0]^T = l_s[\xi_s \ 0 \ 0]^T$. Moreover, $\boldsymbol{\omega}_s = [0 \ 0 \ \dot{\theta}_s]^T$. Then, using Eqs. (5a) and (14), we obtain

$$m_s = \rho_s A_s l_s, \qquad S_{s0} = m_s l_s [s_{s0} \ \ 0 \ \ 0]^T$$
$$J_{s0} = \text{diag}\, m_s \left[\frac{1}{8} d_s^2 \quad \frac{1}{16} d_s^2 + l_s^2 j_{s0} \quad \frac{1}{16} d_s^2 + l_s^2 j_{s0}\right]$$
$$\tilde{\Phi}_s = \text{block-diag}\, m_s [\mathbf{0}^T \quad \tilde{\boldsymbol{\phi}}_s^T \quad \mathbf{0}^T] \tag{83}$$
$$\bar{\Phi}_s = m_s l_s \begin{bmatrix} 0^T & 0^T & 0^T \\ 0^T & 0^T & 0^T \\ 0^T & \tilde{\phi}_s^T & 0^T \end{bmatrix}, \qquad \hat{\Phi}_s = 0$$

in which A_s is the area, l_s the length, and d_s the diameter of the beam, and

$$s_{s0} = \int_{\xi_l}^{\xi_r} \xi_s \, d\xi_s, \qquad j_{s0} = \int_{\xi_l}^{\xi_r} \xi_s^2 \, d\xi_s$$
$$\tilde{\boldsymbol{\phi}}_s = \int_{\xi_l}^{\xi_r} \boldsymbol{\phi}_s \, d\xi_s, \qquad \tilde{\boldsymbol{\phi}}_s = \int_{\xi_l}^{\xi_r} \xi_s \boldsymbol{\phi}_s \, d\xi_s \tag{84}$$

where for beam 1, $\xi_l = -0.5$, $\xi_r = 0.5$ and for beams 2 and 3, $\xi_l = 0$, $\xi_r = 1$. In addition,

$$M_s = m_s \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{H}_s = 0$$
$$\bar{H}_s = -m_s \dot{\theta}^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad K_s = \frac{EI_s}{l_s^3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{85}$$

in which

$$M_{yy} = \int_{\xi_l}^{\xi_r} \boldsymbol{\phi}_s \boldsymbol{\phi}_s^T \, d\xi_s \tag{86a}$$
$$K_{yy} = \int_{\xi_l}^{\xi_r} \frac{d^2\boldsymbol{\phi}_s}{d\xi_s^2} \frac{d^2\boldsymbol{\phi}_s^T}{d\xi_s^2} \, d\xi_s \tag{86b}$$

To test the computer program, a bang-bang torque is applied to the center beam, which is given by $M = 100$ Nm,

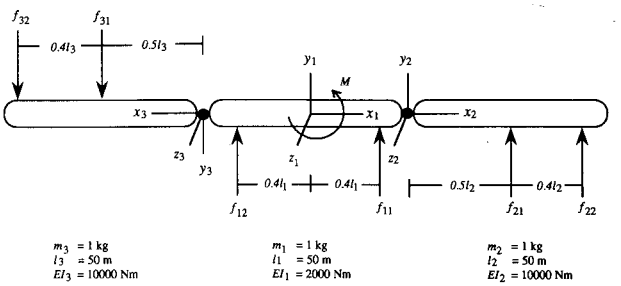

Fig. 4 Three-beam system.

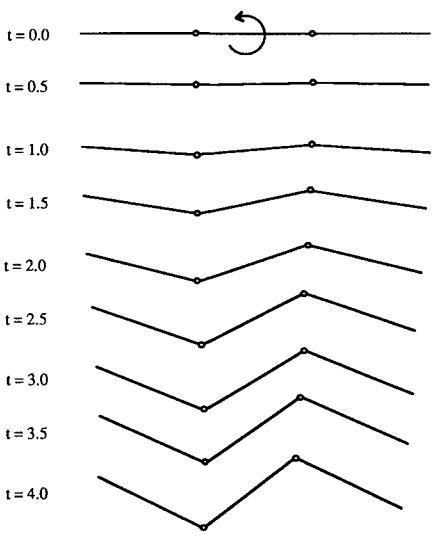

Fig. 5 Time lapse picture of the three-beam system.

$0 \le t \le 2$ s and $M = -100$ Nm, $2 \text{ s} < t \le 4$ s. The control law for the actuators in charge of the suppression of vibrations and perturbations in rigid-body motions during maneuvering is direct velocity feedback control, or

$$f_{si} = -0.5\dot{u}_{sy}(r_{si}, t), \qquad s = 1,2,3, \qquad i = 1,2 \tag{87}$$

where u_{sy} is the y_s component of $\boldsymbol{u}_s$ (i.e., the bending displacement), and r_{si} represents the location of each actuator. The body axes for the various beams are as shown in Fig. 4. Because the applied torque excites only the antisymmetric motion, it appears natural to represent the elastic motion of the center beam by antisymmetric shape functions. The shape functions are chosen from the space of quasicomparison functions[13] and represent linear combinations of clamped-free and clamped-pinned shape functions for each half of the beam. For the peripheral beams, linear combinations of clamped-free shape functions were used. Clamped-free and clamped-pinned shape functions are given by

$$\phi_s = \cosh \lambda_i z - \cos \lambda_i z - \sigma_i(\sinh \lambda_i z - \sin \lambda_i z) \tag{88}$$

where $\lambda_i = 1.875, 4.694, 7.855, 10.996, 14.137, \ldots$, $\sigma_i = 0.734, 1.018, 0.999, 1.000, 1.000, \ldots$ for the clamped-free shape

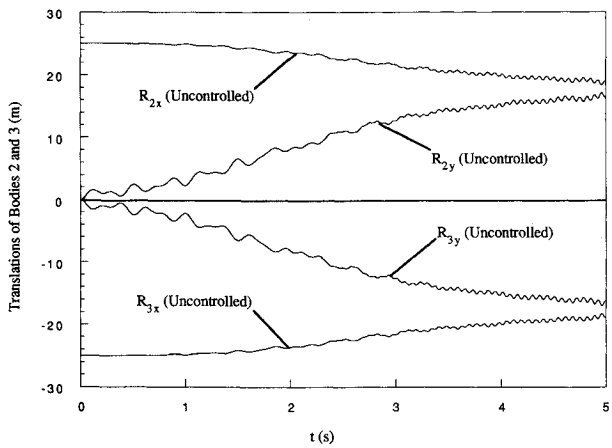

Fig. 6 Time histories of the translational motions of beams 2 and 3.

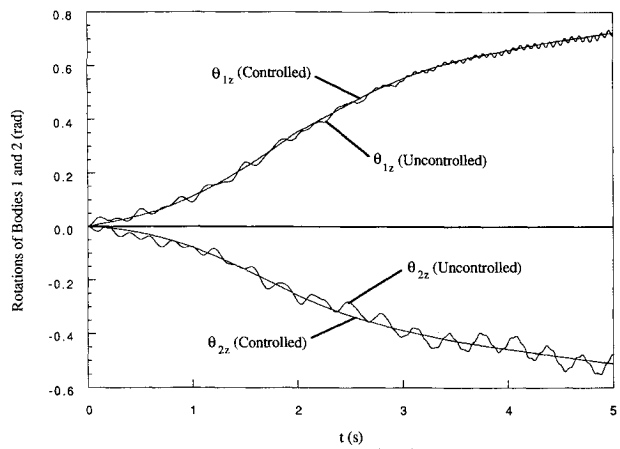

Fig. 9 Time histories of the rotational motions of beams 1 and 2.

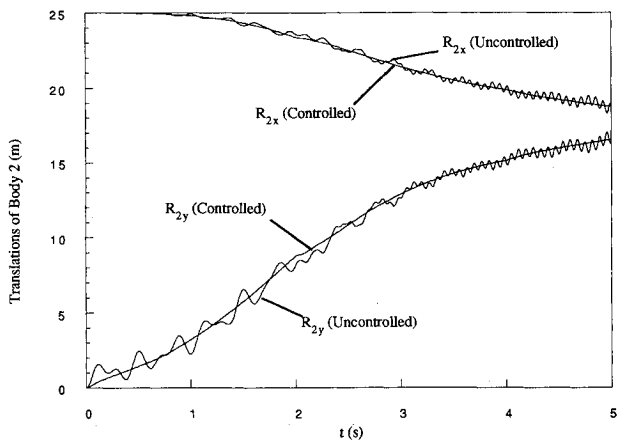

Fig. 7 Time histories of the translational motions of beam 2.

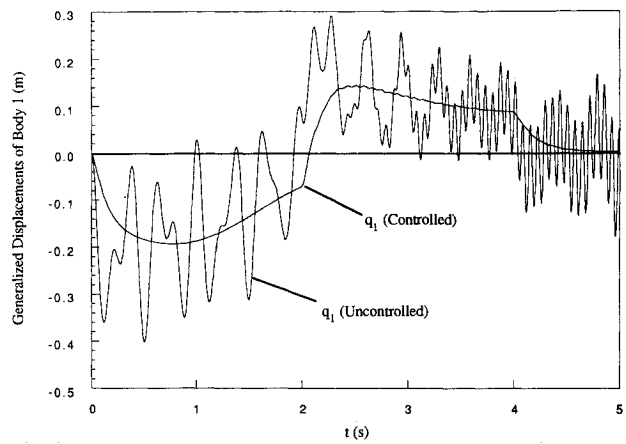

Fig. 10 Time histories of the generalized displacements of beam 1.

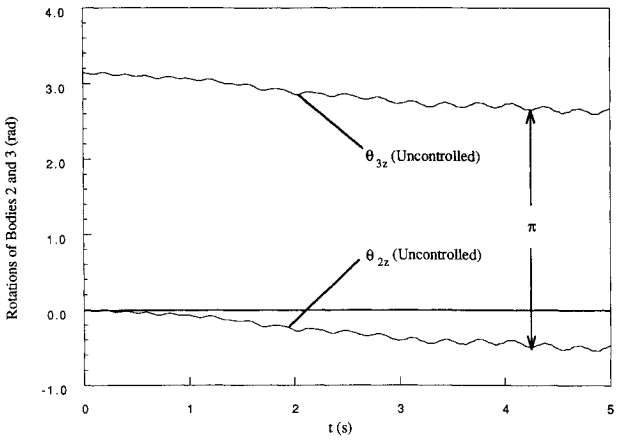

Fig. 8 Time histories of the rotational motions of beams 2 and 3.

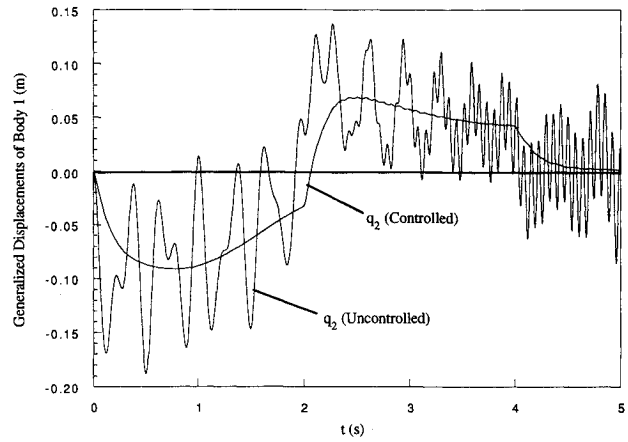

Fig. 11 Time histories of the generalized displacements of beam 1.

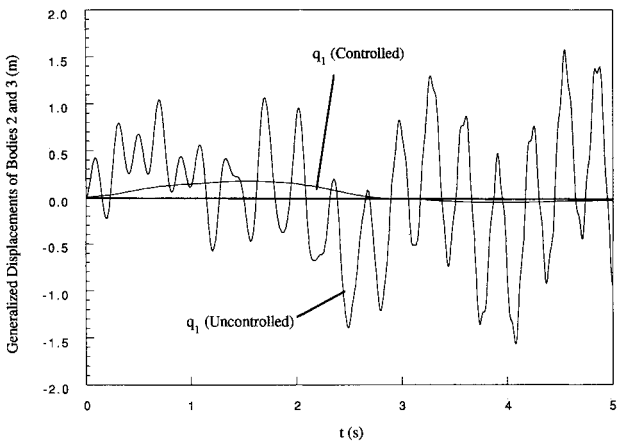

Fig. 12 Time histories of the generalized displacements of beams 2 and 3.

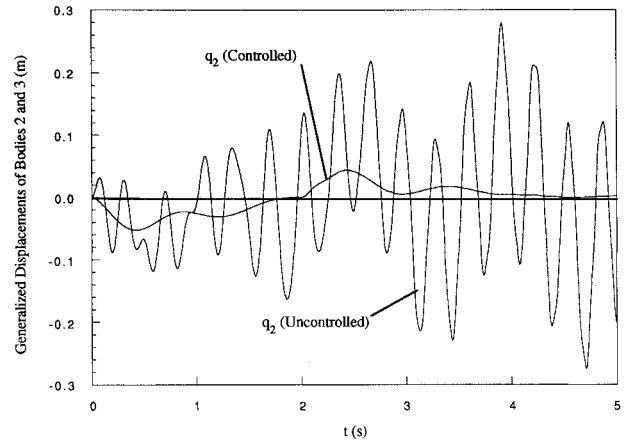

Fig. 13 Time histories of the generalized displacements of beams 2 and 3.

functions and $\lambda_i = 3.927, 7.069, 10.210, 13.352, 16.493, \ldots$, $\sigma_i = 1.001, 1.000, 1.000, 1.000, 1.000, \ldots$ for the clamped-pinned shape functions.

Figures 5–13 show time histories of the response of the system. The time integration of state equations was carried out by means of the IMSL routine DIVPAG. Figure 5 shows a time lapse picture of the three-beam configuration based on the zero-order solution. Figures 6–13 show time responses of the system, i.e., combinations of the zero-order and first-order solutions. Because of symmetry, the middle of the center beam does not move and joints 2 and 3 move antisymmetrically, as shown in Fig. 6. Figure 7 shows the time history of the uncontrolled and controlled translational motions of beam 2. The uncontrolled response clearly exhibits the effect of flexibility. On the other hand, the controlled response is much smoother, reflecting the fact that the elastic vibration and the perturbations in the rigid-body motions have been suppressed. The response of beam 3 exhibits exactly the same characteristics as the response of beam 2. Moreover, from Fig. 8, we conclude that beams 2 and 3 rotate in opposite sense, so that they maintain an angle difference equal to π at all times. Figure 9 shows the time history of uncontrolled and controlled rotational motions of beams 1 and 2. Based on Figs. 7 and 9, we can conclude that the direct-velocity feedback control law used in this example, which is based on the velocity of elastic motions, is capable of suppressing perturbations in rigid-body motions. Figures 10–13 show time histories of uncontrolled and controlled generalized coordinates corresponding to the bending vibration of each beam. It is shown that the direct-velocity feedback control law is effective in suppressing vibrations. Because of symmetry, the elastic vibrations of beams 2 and 3 are exactly the same during the maneuver.

IX. Summary and Conclusions

This paper is concerned with the development of a mathematical formulation capable of treating the problem of maneuvering and control of flexible multibody systems. The formulation is based on equations of motion in terms of quasi-coordinates derived for each substructure independently of the other substructures. The individual substructures are made to act as a single structure by means of a consistent kinematical synthesis. The net effect of this synthesis is the elimination of the redundant coordinates resulting from the original treatment of the substructures as if they were independent.

In most problems of interest, the elastic motions are small compared to the maneuvering motions. In recognition of this, a perturbation approach is developed whereby the rigid-body maneuvering of the system defines the zero-order problem and the elastic motions and perturbations from the rigid-body maneuvering define the first-order problem, where the term "order" refers to magnitude in a perturbation sense. The kinematical synthesis mentioned earlier is applied to both the zero-order problem and the first-order problem. The zero-order problem for the rigid-body maneuvering is nonlinear and of relatively low dimension and can be solved independently of the first-order problem. The control can be open loop or closed loop. On the other hand, the zero-order solution induces time-varying coefficients and persistent disturbances in the first-order problem. The system of state equations describing the first-order problem is linear, time varying, and of high dimension. The equations are in terms of components about body axes, which makes them ideally suited for control design. The control can be divided into two parts: one closed-loop part designed to control transient disturbances and another part designed to reject persistent disturbances. The formulation lends itself to ready computer implementation.

As a numerical example, a planar three-beam system is maneuvered according to a bang-bang control law by means of a torque applied to the center beam. Each beam is equipped with two actuators in charge of suppressing vibrations and perturbations in rigid-body motions; they exert forces according to the direct velocity feedback control law. The numerical example shows the effectiveness of the algorithm developed in this paper as well as the applicability of the equations to the maneuvering and control of flexible multibody systems.

Acknowledgments

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References

[1]Meirovitch, L., and Hale, A. L., "Synthesis and Dynamic Characteristics of Large Structures with Rotating Substructures," *Proceedings of the IUTAM Symposium on the Dynamics of Multibody Systems,* edited by K. Magnus, Springer-Verlag, Berlin, 1978, pp. 231–244.

[2]Hollerbach, J. M., "A Recursive Lagrangian Formulation of Manipulator Dynamics and a Comparative Study of Dynamics Formulation Complexity," *IEEE Transactions on Systems, Man, and Cybernetics,* Vol. SMC-10, No. 11, 1980, pp. 730–736.

[3]Huston, R. L., "Flexibility Effects in Multibody System Dynamics," *Mechanics Research Communications,* Vol. 7, No. 4, 1980, pp. 261–268.

[4]Huston, R. L., "Multi-Body Dynamics Including the Effects of Flexibility and Compliance," *Computers and Structures,* Vol. 14, Nos. 5–6, 1981, pp. 443–451.

[5]Book, W. J., "Recursive Lagrangian Dynamics of Flexible Manipulator Arms," *International Journal of Robotics Research,* Vol. 3, No. 3, 1984, pp. 87–101.

[6]Yoo, W. S., and Haug, E. J., "Dynamics of Articulated Structures. Part I. Theory," *Journal of Structural Mechanics,* Vol. 14, No. 1, 1986, pp. 105–126.

[7]Yoo, W. S., and Haug, E. J., "Dynamics of Articulated Structures. Part II. Computer Implementation and Applications," *Journal of Structural Mechanics,* Vol. 14, No. 2, 1986, pp. 177–189.

[8]Changizi, K., and Shabana, A. A., "A Recursive Formulation for the Dynamic Analysis of Open Loop Deformable Multibody Systems," *Journal of Applied Mechanics,* Vol. 55, 1988, pp. 687–693.

[9]Shabana, A. A., "On the Use of the Finite Element Method and Classical Approximation Techniques in the Non-Linear Dynamics of Multibody Systems," *International Journal of Non-Linear Mechanics,* Vol. 25, No. 2, 1990, pp. 153–162.

[10]Meirovitch, L., "Hybrid State Equations for Flexible Bodies in Terms of Quasi-Coordinates," *Journal of Guidance, Control, and Dynamics,* Vol. 14, No. 5, 1991, pp. 1008–1013.

[11]Meirovitch, L., and Kwak, M. K., "State Equations for a Spacecraft with Maneuvering Flexible Appendages in Terms of Quasi-Coordinates," *Applied Mechanics Reviews,* Vol. 42, No. 11, Pt. 2, Nov. 1989, pp. S161–S170.

[12]Meirovitch, L., *Computational Methods in Structural Dynamics,* Sijthoff & Noordhoff, The Netherlands, 1980.

[13]Meirovitch, L., and Kwak, M. K., "Rayleigh-Ritz Based Substructure Synthesis for Flexible Multibody Systems," *AIAA Journal,* Vol. 29, No. 10, 1991, pp. 1709–1719.

[14]Meirovitch, L., and Quinn, R. D., "Equations of Motion for Maneuvering Flexible Spacecraft," *Journal of Guidance, Control, and Dynamics,* Vol. 10, No. 5, 1987, pp. 453–465.

[15]Meirovitch, L., and Quinn, R. D., "Maneuvering and Vibration Control of Flexible Spacecraft," *Journal of the Astronautical Sciences,* Vol. 35, No. 3, 1987, pp. 301–328.

[16]Meirovitch, L., and Kwak, M. K., "Dynamics and Control of a Spacecraft with Retargeting Flexible Antennas," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 2, 1990, pp. 241–248.

[17]Meirovitch, L., and Kwak, M. K., "Control of a Spacecraft with Multi-Targeted Flexible Antennas," *Journal of the Astronautical Sciences,* Vol. 38, No. 2, 1990, pp. 187–199.

[18]Meirovitch, L., and Kwak, M. K., "Control of Flexible Spacecraft with Time-Varying Configuration," *Journal of Guidance, Control, and Dynamics,* Vol. 11, No. 2, 1992, pp. 314–324.

[19]Meirovitch, L., Stemple, T., and Kwak, M. K., "Dynamics and Control of Flexible Space Robots," CSME Mechanical Engineering Forum 1990, Toronto, Canada, June 3–9, 1990.